A CRITICISM

ON THE

LEGENDRE MODE OF THE RECTIFICATION

OF THE

CURVE OF THE CIRCLE.

BY

J. RALSTON SKINNER.

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INTRODUCTION.

IT was discovered by Rev. John Taylor, that the *height* of the pyramid of Jizeh, or Cheops, bore the relation to *twice the base side*, of *diameter to circumference of a circle*. This relation is *typical* of its interior construction, and likewise of the *modulus* of *measure* used in that construction.

Take the numerical form 20612 : 6561. This is, agreeably to the late John A. Parker, of the City of New York, the true integral relation of circumference to diameter of a circle. What is called the *pi* value of this expression is 3.14159426+. The Legendre, or established value, is 3.14159265+. The difference, in all but extraordinary exceptions, is practically, quite inappreciable, — the value used in ordinary engineering is 3.1416.

§ 1. (*a*) Take this form, 20612 : 6561, as $20^{\underline{612}} : 6.^{\underline{561}}$, and consider it as *British inches;* then this $20^{\underline{612}}$ B. inches is the ancient Egyptian *Cubit* value, one of the so-called royal cubits. From measurements of this pyramid by Prof. Greaves, of Oxford, England, Sir Isaac Newton restored the value of this ancient cubit, which, to specialize it, is now called the *Turin cubit*, as 20.604 B. *inches*, or 1.717 B. *feet*. From measures of the catacombs of Osimandya, Egypt, by the French *savans*, under Napoleon, this cubit was restored in terms of the French meter, as .523524, which gives, by reduction to B. *inches*, $20.^{\underline{61156}}$. This last value, *plus* .00044 of the inch, equals $20.^{\underline{612}}$, or the above given value of circumference. We take it, then, that the true restoration of this cubit is $20.^{\underline{612}}$ British *inches;* and that it had its origin from the geometrical source of 20612 : 6561, or of circumference of a circle whose diameter is 6561.

(*b*) Take the form, or proportion—

$20^{\underline{612}} : 6 :^{\underline{561}} : : 64.^{\underline{8}} : 20.^{\underline{6264700174+}}$ as in British *inches*. It is seen that the fourth term is exceedingly near in value to the *first* term; yet it is, indeed, of a very significant difference. (1) While the first term is a *circumference* or curved value, the

fourth term is a *diameter*, or straight line, value; (2) This fourth term is diameter to an integral circumference value, viz., $64.\underline{^{8}}$.

Now, this fourth term, as B. *inches*, is the so-called *Nilometer cubit* value, or the other of the royal cubits. The restoration of this cubit value, by Mr. Wilkinson, was found to be $20.\underline{^{625}}$ B. inches, showing a difference from the above of .0015 of an inch.

Thus we see that these two royal cubits were geometrically related to each other, and referred themselves to the above numerical relation of circumference to diameter of a circle.

(*c*) Take $6.\underline{^{561}}$ out of the above formula, as B. *inches*,—multiply it by $\frac{16}{9}$, and the result will be $11.\underline{^{664}}$ B. inches. This result gives the restored value of the ancient Roman *foot*. An abundance of authorities could be given for this. (See The Great Pyramid, by John Taylor, Chap. III.)

§ 2. The *analytical unit* of measure, so-called, in geometry, is obtained by the formula $\frac{180}{pi}$.

This analytical unit is where the length of a curved line of a circle is the same, numerically, with the lengths of the two radii embracing that curve. By the above form of Mr. Parker, in the circle of 360°—(1), the radii containing the curve; (2), the curve itself; and (3), the angle measuring the curve; all are numerically of the same value, thus—

Value of radius, say in feet............... 57.2957500048+
" of curve, " " ....57.2957500048+
" of angle, in degrees................57°.2957500048+

Here it is seen that *feet*, or any other unit of measure, can be read as *degrees*. The two readings are convertible. Now this *analytical unit* is directly obtainable from the above proportion; for divide 64800 : 20626.4700174+ by 360, and there results 180 : 57.2957500048+, or circumference to diameter. If 57.295+ is taken as *radius*, then 180 must be multiplied by 2, making 360 for its circumference.

(*a*) This is given, because, while the type of the structure of the pyramid is circumference to diameter, the subordinate from which the type is taken is this veritable analytical unit of measure.

[NOTE: The reconstruction of The Great Pyramid is on the basis, that every detail of measure *must* answer to the reliable measures thereof, taken in terms of the present British standard (Kater standard now in Edinburgh). The measures to be relied on are, in the order of time in which they were taken, those of Greaves, of Oxford, the French *savans* under Napoleon, Col. Howard Vyse, of Mr. Lane, and of Mr. Piazzi Smith. All are to be found in "Life and Works at the Great Pyramid," by Mr. Smyth. If, in the reconstruction of the pyramid, the present measures, founded as they are on a geometrical truth, viz., the Parker *modulus*, prove themselves to have been, of very necessity, the ones used by the architect, in the plans of the structure, they must serve as corrective of the trial measures of these gentlemen, in all cases of difference of measures, where the difference is very small indeed, as in hundredths of a *foot*, or *inch*, and in long distances even in tenths of a *foot*.]

§ 3. The mode of origin of the British *foot* upon, or from, the above *modulus*, is this: (1), $20.\underline{^{612}}$ to $6.\underline{^{561}}$ as $64.\underline{^{8}}$ to $20.\underline{^{62647001+}}$ *inches;* (2), $64.\underline{^{8}}$ to $20.\underline{^{62647+}}$ as 12 to $38.\underline{^{1971+}}$ *inches*.

Here in (2), the *third* term derived as seen, is the number of *inches* founding the tables of British *long measures*, in that which is technically denominated as *one foot*. This origin is singular and noteworthy, for the *fourth* term with which it is directly connected, is a foundation measure of the Great Pyramid of Egypt, in the scale of *inches* for *feet:* for $381.\underline{^{971+}}$ *feet*, is the precise length of the half base side of that pyramid.

§ 4. By the above *modulus* of measure the Great Pyramid of Egypt was *architecturally* constructed. The averment can be made positively, as justified by the fact of reconstruction of the mass, and of the interior details. Let us illustrate this by means of the measures of the King's chamber, and of the coffer in that chamber. Having the measures of this chamber, they are typical of the construction of the whole work, from this *modulus*, and the coffer, *as an independent structure*, in its measures, is placed as a check, and proof, and independent verification, of the proper measures, and *modulus* of measures, used in and about the whole construction.

The King's Chamber measures are as follows:

LENGTH.

By Mr. Smyth ....$412.\underline{56}$ inches.
" Mr. Lane..$412.\underline{50}$ "

WIDTH.

By Mr. Smyth ....$206.\underline{28}$ inches.
" Mr. Lane..$206.\underline{25}$ "

HEIGHT.

Mr. Smyth is uncertain of the height, owing to the disturbed state of the floor—become so since Vyse.

By Howard Vyse ....$229.\underline{1}$ inches.
" Aiton and Inglis...................$229.\underline{1}$ to $229.\underline{2}$ "

Take our Nilometer cubit out of the above form, or *modulus* of measure, viz., $20.\underline{62647001+}$ inches. This multiplied by 20 gives $412.\underline{529+}$ inches, or 20 Nilometer cubits, as the length of the room. The width is half the length, or $206.\underline{2647+}$ inches. Multiply $20.\underline{62647+}$ by $\frac{100}{9}$, and there results $229.\underline{1830+}$ inches, which is the height of the room. In length the room is $34.\underline{37745+}$ *feet*. Compare these results with the above stated measures of Vyse and others.

Now to the Coffer. The measures thereof now to be given are all to be found as those made, actually, by Mr. Smyth (see his Works) to within exceedingly close limits (from 1 to 2 *hundredths* of inches).

Parker form of diameter to circumference of a circle 6561 to to 20612. From this we can take the proportion

$$20\underline{612} : 6\underline{561} \text{ as } 64\underline{8} : 20.\underline{62647001+}$$

Call these terms *British inches*, and in the first and fourth terms we have the ancient *royal cubits*, so-called; by which cubits the Great Pyramid was constructed in all its parts.

Take the third and fourth terms as 648 to $206.\underline{2647001+}$ inches; divide by 24, and we have, for circumference and diameter,

$$(1),\ 27 \text{ to } (2),\ 8.\underline{5943625+} \text{ inches.}$$

(1), or 27 inches, is the inside width of the Coffer. The wall of the Coffer is 6 inches thick, therefore $6 \times 2 = 12$ inches added to 27 inches equals 39 inches, the outside width of the Coffer; 39 inches multiplied by 2 equals 78 inches, which equals the inside length of the Coffer.

Multiply (2) or $8.\underline{5943625+}$ inches by 4, and we have $34.\underline{3774500+}$ inches, or inside height of Coffer. This, in the scale of inches for feet, is the length of the King's Chamber. Multiply $34.\underline{3774500+}$ inches by $\frac{12}{10}$, or $20.\underline{62647001+}$ inches by 2, and we have $41.\underline{25294002+}$ inches, which is the outside height of Coffer; $412.\underline{529+}$ inches is the length of King's Chamber.

So we have

Inside width,	27.	inches.
Inside length,	78.	inches.
Inside height,	$34.\underline{3774500+}$	inches.
Outside width,	39.	inches.
Outside length,	90.	inches.
Outside height,	$41.\underline{252940+}$	inches.

Cubic contents of inside of Coffer, $72398.\underline{90097612846+}$ *inches.* Multiply this by 2, and we have the cubic contents of the outside measures, or

$144797.\underline{81952256920+}$ inches.

These measures, as said, compare with, and explain or interpret, Piazzi Smith's actual measures of this Coffer.

From the above, the thickness of the bottom of the Coffer is $6.\underline{8754+}$ inches. This multiplied by $\frac{100}{3}$ is $229.\underline{1830+}$ inches, which is the height of this room. So, also, $68.\underline{754}$ is diameter to a circumference of 216 exactly, which is the cube of 6.

Enough has been given to show that all this measuring duplication, verification, and correction, *can not* arise from accident, but *necessarily*, and architecturally, from the above *modulus* of measure, originating the British inch, as the fundamental origin of the measures of the great controlling nationalities of the world.

And this leads up, directly, to investigation into the merits of the Parker quadrature of the circle, or value of *pi*. I now aver it to be *the*, and the *only*, true one. It will bear every known test, geometrically and otherwise (see "The Crown Jewels of the Nations are their Measures," sec. 1, Geometry). Before, however, this *pi* relation can be accepted, or perhaps even entertained, the established relation, as now to be found in our geometries, and which lays as the base of the higher mathematics, *must be shown to be in essential error*, and it is claimed that this is

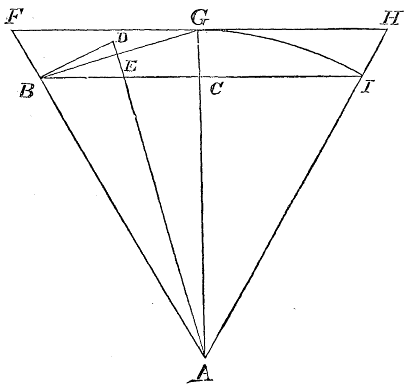
done in this monograph. The truth is, that the numerical results of the accepted mode of obtaining the rectification of the curve of the circle are correct, but the application of these results as having any thing to do with the curve of a circle, or any other curve, is a false statement, and is vicious.

A CRITICISM ON THE LEGENDRE MODE OF THE RECTIFICATION OF THE CURVE OF THE CIRCLE.

THE ESSENTIAL OF THE LEGENDRE PROBLEM.

I.

§ 1. (*a*) The Legendre mode of calculating π is strictly by use of isosceles triangles, without any relative structural, geometrical use, or any numerically measuring use, of the circle whatever. A B C is a right-angled triangle. Extend A C until



A G = A B, and join B G. A B and B C are known, and $\sqrt{AB^2 - BC^2} = AC$. Then $AB - AC = CG$, and $\sqrt{BC^2 + CG^2} = BG$. Draw A D (= A B) bisecting B G, and join B D; then proceed with similar computations as far as desirable. Any ultimate triangle, say A B D, will be an even known part, or factor, of the entire regular polygon. The leg A B is said to be the *radius* of the polygon. To a radius of $\frac{5}{10}$, ultimately, the side $BD \times n$ (or total number of sides to make the polygon), will be found to equal 3.1415926535897+.

This, strictly, and in its simplicity, is, inclusively and exclusively, all that there is of the Legendre mode of obtaining the so-

called established value of π; and, as seen, its object is to obtain the numerical value of the straight line B D, when that has become exceedingly small. Let the numerical value of the straight line B D be a, and let n be the number of sides of the polygon; then, the formula for the ultimate expression (where the leg, or *radius*, A B of the isosceles triangle is $\frac{5}{10}$) will be

$$a \times n = 3.14159265358 97 +.$$

(b) Beyond this, but not in any manner an essential part of the problem, we have A C : C B = A G : G F, where C B is $\frac{1}{2}$ of side of the interior, and G F is $\frac{1}{2}$ of side of exterior, polygon.

It is alleged, that by the working of the problem G F finally becomes at one with, and the same as, C B; and the use of this G F, or side of the exterior polygon, is to show that its value is *a limit* on the possible excess of the value of the curved line of the circle I G B over the chord I C B. But from the essential nature of the problem, this phase of it is wholly inapplicable; and it is, in fact, a deceit and illusory.

The numerical value of π in this problem is obtained, as stated, exclusively by calculation of the sides of the interior polygon. The result is *numerical*, not *geometrical*. The numerical result is the measure of a polygon not of a circle. To find the true application of this result to the admeasurement of a circle, *the polygonal area must be converted into a circular one*. By doing so, it will be found that both radius and circumference have to be taken into consideration, in this respect, viz., that the radius of the equivalent circle can not be represented by *any* line of the polygon, nor the circumference by its perimeter. Radius of the equivalent circle will inevitably be shorter than that of the circle penned up between the polygons, and the length of circumference will be less than that of the same circle. The exterior polygon, then, has no such function of limitation as is claimed for it. The reasoning applies to any possible condition of reduction.

THE NATURE OF THE EXPRESSION $a \times n = 3.1415926535.+$

II.

By the above [1. (a),] — and the above in its bare simplicity is all there is of the Legendre problem—it is seen that there is nothing whatever beyond the use and calculation by *numbers*, of *straight* lines of isosceles triangles. Let us ascertain the nature

and meaning of the ultimate expression sought for, and obtained, viz.:

$$a \times n = 3.1415926535+.$$

At 6144 sides of the polygon, whose leg, or so-called *radius*, A B, is $\frac{5}{10}$, the value of the straight line B D is .00051132690 7013+, and $a \times n = 3.1415925166+$. Here B D is obtained as $\sqrt{BE^2 + ED^2}$; that is: (1), B D is a *straight line*; (2), its value is obtained as a $\sqrt{}$, or *square root*, and (3), the numerical expression of its value is what is called *incommensurable*, or *irrational*. Technically, expressions of this kind are called *surds*, and the value of B D is precisely a *surd* expression of the quality of quantity, just as is $\sqrt{2}$ or $\sqrt{8}$. If B D is a *surd* expression, its multiples must be *surds*, and $a \times n$, or 3.1415926535+ is a *surd* expression of the numerical value of a number of straight or right lines, but, by no possibility, of any curve whatever. This applies in exactly the same way, and after the same fashion, to any, so taken, ultimate expression, as 3.1415926535+.

We can suppose *a limit* to be obtainable, by which a *surd* expression will become rational; or, by which, by way of illustration, the value of B D as .000511326+ may be converted into a determinate decimal; or, in other words, by which the exact, instead of the inexact, numerical value of the straight line B D may be had.

Let us suppose this to be the case: All that has been attained, by the conversion of the irrational into a rational expression, has been *the exact numerical value of the straight line* B D, the side of the isosceles triangle A B D. The multiplication of this value will never alter the nature of it as presented. The multiplication of length of a straight line is a straight line in its very nature. It follows from this that the expression $a \times n =$ 3.1415926535+, is (1), that of the multiplied value of *a straight line*, (2), that it is a *surd*, and (3), that if the limit could be had for its conversion into a rational or determinate quantity, its rational value would only amount to the exact expression of numerical value of the straight line B D, and its multiples.

(*a*) This conclusion leads directly to a result of very great consequence. The ultimate straight line B D, side of the isosceles triangle A B D, is *chord of the arc* of part of the circle whose *radius* is A B. The length of the curve of an arc of a circle must be in excess of the length of its chord; therefore, if a is the length of the chord, the expression for the length of the curve of

which it is the chord, must be $a + x$. If n be the number of sides to compose the polygon, then the expression for the total circumference of the curve of the circle will and must be

$$\overline{a + x} \times n.$$

(*b*) Let y (to make the matter quite clear) be the limit in value, by which a may be converted from an irrational to an integral form: we will have for the possible, rational and exact value of B D, $\overline{a + y}$; and for the entire curve we will have the formula,

$$(\overline{a + y} + x) \times n.$$

Where of this formula, $a \times n = 3.1415926535+$. Great efforts have been made to rationalize this *surd* expression of $3.1415926+$, but should they ever prove successful, the result would, after all, only apply to the straight line B D and its multiples, and not at all to the curve of which B D is the chord.

(*c*) Comparatively, therefore, where the radius or leg, A B, is $\frac{5}{10}$, we have the *formulæ:*

(1) For the perimeter of inscribed polygon, $a \times n = 3.1415926535+$.

(2) For circumference of circle $\overline{a + x} \times n =$?.

THIS VALUE OF INCREMENT x NOT RECOGNIZED BY THE LEGENDRE MODE.

III.

This formula, $a + x \times n$, which is so pure, so simple, and so true, as the expression of the value of the curved line, is never to be met with in geometries; use of it is avoided by a subtlety. As said, the expression for the ultimate value of perimeter of inscribed polygon is $3.1415926535+$, being $\overline{a \times n}$. Now, it is claimed that, upon ultimate reduction of the sides of the inscribed and circumscribed polygons, the numerical value of the exterior polygon has, in effect, approached to, and become in common with, and the same as, that of the interior, that is: F H has become equal to B I. This could not be the case unless geometrically, by gradual approach, the line F H had assimilated with, and become in common with, B I. Such being the case, the curved line I G B, penned up between these sides, must likewise have become in common with the others as to shape, extension, and value. Consequently, no such value as x in excess can be

entertained, and the values of F H and the curve I G B have vanished into, and become at one with, that of B I. However absurd it seems, this is *a necessity* of the Legendre mode, and is purposed to enforce this peculiar and seemingly inevitable restriction upon construction, viz.:

If the curved line of the circle be held as in excess of the perimeter of the polygon, that excess can only amount to, and be limited by, the difference in excess of the value of the sides of the exterior, over that of the sides of the interior, polygon; and this difference of excess, being shown ultimately to vanish, leaving the value of the sides of the exterior, equal to that of the sides of the interior, polygon, ultimately, the curved line of the circle can have no excess in value over and above the length of the perimeter of the inscribed polygon.

COMPARATIVE DEFINITIONS OF A CIRCLE, AND THE REASON FOR THE DIFFERENCE.

IV.

The structure of the circumference line of a circle is such that the definition thereof is essential in itself, and has no need of exterior, or extraneous, conditions. The definition is one that follows strictly the mental conception of the structure of such a curve, and likewise the mechanical fact: *The curve of a circle is of such a nature, that as to any, even the least part thereof, if such part be protracted either way, it will finally re-enter on itself and form the entire circumference of the circle.* By the foregoing mode of Legendre, it is alleged that the area and shape of the interior polygon, at the point of the ultimate reduction of its straight-line sides, becomes the area and shape of a circle; so that, as a circle, and as a polygon, the two shapes, upon application, the one to the other, will prove to be one and the same shape. For this reason, the above definition of a circle is never found in this connection, in geometries, but by subtlety, and pious fraud, another definition has been framed to fit the awkward situation of conditions, as follows: *The curve of a circle is of such a nature that every point of the circumference is equally distant from its center;* where the *point* is construed as the ultimate straight-line side of the polygon. As might be expected with regard to such forced, arbitrary, terms of definition, and modes of construction, geometry is not at all times at one with itself, for as to obtaining the area of a polygon, the rule

is sometimes, and unwarily, laid down; — $\frac{1}{2}$ *perimeter of polygon by radius of inscribed circle.* As further commentary, the protraction of the side of a polygon is indefinite extension of a straight line, and is necessarily tangent to, instead of following, the curve of the circle.

The fact is, that exclusive use has been made of triangular figures, made up of right lines, and their numerical admeasurements, by this mode of Legendre; then the whole matter, *as to terminology*, has been transferred over into another realm, as nomenclature of the geometrical shape of the circle and its relative parts, whose elements of structure are utterly unknown. Then, this terminology is made applicable as a right use and nomenclature of the elements of circular shapes and their numerical admeasurements.

STRUCTURAL DIFFERENCES OF CIRCLE AND POLYGON ARE ESSENTIAL.

§ 2. Impossible as it may seem, there appears to have been in the minds of Legendre and Playfair no contemplation of the *essential structural differences* of the two different kinds of shapes, where the one is structurally, or mechanically, sought to be converted into the other, for an equal area.

A.

In the case presented in § 1, we have the area of a polygon composed of an infinite number of isosceles triangles, and of an infinite number of sides, which *is called*, but *is not*, the area of a circle.

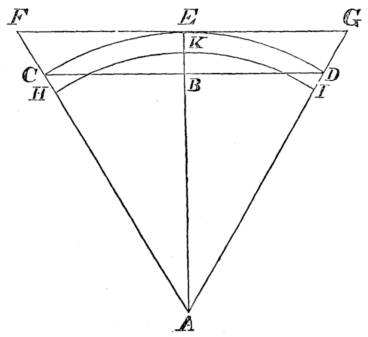
(1) Take a circle and a polygon, whose perimeters have the same length; then, the polygon will have less area than the circle, and a similar polygon having the same area with the circle, will have a greater length of perimeter. Hence it follows, that if any one of the isosceles triangles spoken of, making up the polygon which is said to become the circle, is structurally, or mechanically, converted into a similar section of a true circle, as a result, there will not be a single line or admeasurement of the one figure, in common with that of the other, in whole or in part, either as to shape or linear measure. As shapes, if applied the one to the other, they will prove wholly incongruous.

(2) The late John A. Parker, of the city of New York, demonstrated structural relations between the circle and polygon, that

effectually prove a specific difference in the structures of the shapes; he demonstrated at the same time a true, exact measuring relation between the shape of a square, or other polygon, and the circle. Let the circle (A) have any length of circumference, and let the square (B) have the same length of perimeter. Convert the area of this square into that of the circle (C). Then the square (D) circumscribed about the circle C, will have the area of the circle A.

B.

The numerical values of the polygons are not indicative of the circle penned up between them, and the exterior polygon has not the function of limiting the excess of the length of the curve of the circle over the length of the perimeter of the inscribed polygon.



Let A C D be the isosceles triangle, 6144 of which complete the polygon. At this, or some more remote, stage of reduction, by geometry and geometries, the triangular area A C D is said to be the circular segment A C E D. The straight-line C D of the triangle is said to be the curved line C E D.

As to measures: As an isosceles triangle, the measure of the area A C D should be $\frac{CD}{2} \times$ A B the perpendicular. But here, although we still have only the shape A C D and the measures of the lines, A B, A C, and C D, of the triangle A C D, to work with, the formula for area is by the mode of Legendre, now changed to $\frac{CD}{2} \times$ A C, which is absolute error; and now this last is called the area of the circular segment A C E D. The falseness of the statement is only equaled by the juggling suppression of all apology in making it.

It must be steadfastly kept in mind that our only measure of the curve of the circle is in the *numerical terms* of measure, of

the interior isosceles triangle A C D, and of that only and alone, as already stated.

(1) Let us take the true polygonal measure of the isosceles triangle A C D of the interior polygon, at its reduction to 6144 sides, where A C $= \frac{5}{10}$. Its side C D is .000511326907134+. Its perpendicular A B is .49999993463+. Its area, a, equals $\frac{\text{C D}}{2}$ $\times$ A B $=$ .00012783171004063+. Then the area of the polygon will be $\overline{a \times 6144} =$ .785398026489+. The area of the polygon at a still very remote reduction of its sides is, by hypothesis, called a circle, whose radius is $\frac{5}{10}$ and circumference 3.1415926535+. By the forced formulation $(\frac{\text{C D}}{2} \times \text{A C}) \times$ 6144, the area of a *circular* segment, similar to the above, of a circle whose radius is $\frac{5}{10}$ and circumference is 3.1415926535+, becomes .78539816339+. To convert the area of the triangle, or segment of the polygon, into that of its similar and equivalent circular shape, or similar segment of a true circle, we have the rule, *that the areas of circles are as the squares of their radii.* Hence .24999995642+ is the square of the radius of the equivalent circle, and its radius is .4999999564+. This radius is seen to be greater than A B, the perpendicular of the triangle, and less than $\frac{5}{10}$, the value of A C of the triangle, called by Legendre the radius of the circle. The elements, then, of this smaller circle are:

Radius, .4999999564+; circumference, 3,141592379O5+.

(2) But we have made this conversion in terms of the π value made up of the sides (each equal to B D, § 1) of the ultimate polygon, or by the formula given in § 1 (a), viz., $a \times n$. As said, the value x, of excess of curve over its chord, is considered as vanishing, is never used in formulating, and is, at any rate, as alleged, *limited* by the measure of the exterior polygon. Here it is seen that the measure of the exterior polygon is *no limitation whatever* on the value of x in this smaller circle, which is the true, and only, circular equivalent of area or volume of the interior polygon. If this is so, and it is so, x is an essentially existing value, absolutely necessary to the solution of the problem, at whatever reduction of the sides of the polygon; it does not vanish, and is not affected nor measured by the circumscribed polygon, so that the question of the value of this excess of x is an open one, and not at all subject to the limitation usually imposed of the exterior polygon. The true formula, then, *is* $\overline{a + x} \times n$.

C.

From this condition we can derive *the true status of limitation* of the radius and circumference of the circle, equivalent in area or volume to the inscribed polygon.

In reducing the sides of the interior polygon to any stage, no matter how remote, the radius of the *equivalent circle* will, in length, *always* be found *to be less* than A B and *greater* than A C (§ 1). Hence, the ultimate polygon will inevitably indicate *a circle whose limit, for radius, will be less than that of the circle penned up between the polygons.* In other words, the geometrical limit of the circle thus obtained makes it, inevitably, less than the penned-up circle, both as to radius and circumference. Hence, it follows that when the ultimatum of reduction is made as to the interior polygon, and this *is called* a circle, then for the measure of the penned-up circle, greater length of radius is to be given, and, consequently, greater length of circumference, than that of the true circle, the equal in area to this ultimate polygon.

(1) But this is by working agreeably to the formula $a \times n$, which involves the measures of the interior polygon, only and alone. In addition to this, the true formula for the circle being $\overline{a + x} \times n$, where x is the excess of length of the curve over its chord, the additional length being (or having been) ascertained for radius, to make it $\frac{5}{10}$, and which can always be had (it depending merely on the stage of reduction), then, to the circumference proportionally indicated by this enlarged radius, the increment x has to be made. In this instance, add to radius thus: $.4999999564 + .0000000436 = \frac{5}{10}$, and circumference becomes, say, $3.1415926535+$. This circumference is $a \times n$. For $\overline{a + x} \times n$, we will have this $3.1415926535+$, *plus* the increment $x \times n$. This increment is one separate from, and in addition to, this $3.1415926535+$, even if this last could be made an integral, or rational, expression. Neither is the value of this increment *affected at all* by the alleged limitation of the exterior polygon, which, in truth, *has no relation to the matter*. It may be remarked, also, that, though at some very remote reduction x may, as is alleged, become exceedingly small, yet its multiplier becomes exceedingly great as it (x) decreases; for while in the original polygon, say of six sides, the multiplier is 6, it becomes successively 12, 24, 48, 96 * * * 6144, and so on, to get the final value, $3.1415926535 + \overline{x \times n}$.

D.

We can find, likewise, the similar polygon, which will be equal in area to the circle penned up between the polygons. Its radius A B (§ 1) will be .500000038+, its side B I will be .0005113269458—, and its perimeter will be 3.14159275499+. This again shows that the exterior polygon is no true limitation on the true measure of the circle penned up as stated. The reason for this is that there are two necessary limitations on the elements of the circle, functional to each other, when compared and equalized with those of a polygon. To obtain area, radius and circumference must be used, and the limitation, to be effectual, must geometrically include and control both these elements. If $a \times n$ gives a circle, it must be said to be of a wholly different nature from that under the true formula $\overline{a + x} \times n$.

CONCLUSION.

§ 3. From the foregoing we can gather the following, as to the Legendre mode:

(1) That, divested of all verbiage and false showings, his process is of the simplest nature, viz., the reduction, by the process shown in § 1, of a single polygon to a very great number of sides, or straight lines, and calling the numerical value of the perimeter of the ultimate polygon that of the circumference of a circle. This it is not;—nor is it, structurally, the measure, even by way of approximation, of the circle penned up between the polygons.

(2) When he has arrived to what he may call his circle, then, to obtain the value of circular area, he abandons use of the perpendicular A C (§ 1), which, with $\frac{1}{2}$ perimeter, is the only geometrical mean of measure of the same, and in its place adopts the use of the leg A B of the isosceles triangle as that mean—a proceeding wholly unwarranted. In place of $\frac{BI}{2} \times AC$, the true and only formula, he *assumes* $\frac{BI}{2} \times AB$, which is manifest error. (See § 1.)

(3) By the usual *theorem*, the statement is made of a reduction of the sides of the inscribed polygon, until the same may be taken, to within less than any assignable value, as the measure of the circle. What circle ? The *theorem* is forced by the nature of the geometrical diagram used, to explain and determine this

"What circle?" by indicating *one especial line.* On this indicated line rests the whole problem for integrity and soundness. If it is falsely taken, the *theorem* fails. The line thus indicated is A B in § 1, as that essential part of the exhibited circle which the polygon is professed to measure, viz., its *radius.* This *it is not*, for the area of this polygon, when converted into the area of a circle, will, *always and inevitably*, show a radius *less* than A B. The *theorem* assumes this line as, in its integrity, belonging *in common*, both to the polygon and to the circle attempted to be measured by this polygon, and if this assumption is untrue the theorem fails. The assumption *is* untrue.

Thus, at the very inception of the problem, we have the two essential elements of the circle, *unknown*, viz., radius or A B, § 1, and circumference, that is, say, x and y. As x and y these are functional to each other, and the one varies in value with any change of value of the other. But this radius, or x, is falsely taken as a known quantity, say a, and to it is given the constant value of $\frac{5}{10}$. Above it is shown, that, relatively to any measure of the circle, it must inevitably be less than $\frac{5}{10}$, and this necessarily imports an increase in the value of y over that usually assigned to it. If x is less than the value assigned to it, then y must be greater, and the ratio of x to y must be different from that usually given. To a radius of $\frac{5}{10}$, circumference must be greater than 3.1415926+.

(4) In all his formulations, he makes no use of x as the excess of the value of the curve over that of its chord. In this he has erred, because that which he claims as limiting this excess has, in truth, no such function of limitation.

CINCINNATI, *Christmas*, 1879.

APPENDIX.

In reduction of the polygons, let us rest at some very remote reduction, and call the polygon a circle, after the Legendre fashion. The measure of this can, hypothetically, be used, relatively, as that of a circle, for any polygon of *a fewer number* of sides—say, from 6 to 6144, inclusive—to show the excess of value of the curve over its chord (the side of the interior polygon). Call this excess $L. x$, or "Legendre excess." If this excess is calculated continuously, it will, of course, be found to grow less